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Symmetries and invariant solutions of the two-dimensional variable coefficient Burgers equation

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Abstract

We discuss symmetries and reductions of the two-dimensional Burgers equation with variable coefficient. We classify one-dimensional and two-dimensional subalgebras of the Burgers symmetry algebra which is infinite-dimensional into conjugacy classes under the adjoint action of the symmetry group. Invariance under one-dimensional subalgebras provides reductions to lower-dimensional partial differential equations. Further reductions of these equations to second order ordinary differential equations are obtained through invariance under two-dimensional subalgebras. The reduced ODEs are then analysed and shown that they belong to the polynomial class of second-order equations which can be linearized only for particular values of parameters figuring in the coefficient.

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1. Introduction

The Korteweg–de Vries–Burgers (KdV–Burgers) equation

$$(u_t + uu_x + \mu u_{xxx} - \nu u_{xx})_x + \sigma u_{yy} = 0 \tag{1.1}$$

is a prototype example of an evolution equation in 2+1 dimensions which is not completely integrable. Here μ, ν are real constants and $\sigma = \pm 1$. Although it has an infinite-dimensional symmetry algebra it does not have a Virasoro structure. The presence of a Virasoro algebra generally signals integrability for (2+1)-dimensional evolution equations. Travelling wave solutions of the two-dimensional KdV–Burgers equation have been studied in a series of papers [1–3].

We restrict ourselves to the two-dimensional generalized Burgers equation

$$(u_t + uu_x - u_{xx})_x + s(t)u_{yy} = 0. \tag{1.2}$$

Equation (1.2) with $s =$ constant is sometimes referred to as the Zabolotskaya–Khokhlov equation in nonlinear acoustics [4, 5]. Painlevé analysis of the constant coefficient version of (1.2) was carried out in [6]. The authors showed that the equation possesses the conditional

Painlevé property and obtained its exact solutions by use of truncation. For $\nu = \mu = 0$, the conservation laws and Lie symmetries of equation (1.1) have been investigated in [7] (see also [8] for a comment).

In this paper we investigate the symmetry group of the variable coefficient partial differential equation (1.2); that is, the Lie group G of transformations acting on the variables (x, y, t, u) taking solutions into other solutions such that whenever $u = f(x, y, t)$ is a local solution of (1.2), then the transformed function $\tilde{f} = gf(x, y, t)$ is a solution for all $g \in G$. Next we proceed to construct group invariant solutions. A systematic study of group invariant solutions requires a classification of the subalgebras of the symmetry algebra into conjugacy classes under the adjoint action of the symmetry group. Here we undertake this task to classify reductions and thereby exact explicit solutions of equation (1.2). The similar steps, except for an analysis of the reduced ODEs, have been applied to the study of symmetry properties of a variable coefficient KP equation in [9]. For a detailed analysis of the symmetry reduction for the usual KP equation using a loop algebra the reader is referred to [10]. We mention that some similarity solutions of the constant-coefficient Burgers equation have already been discussed in [11], but the authors chose subgroups for performing a reduction by restricting the arbitrary functions in the symmetry group to first-degree polynomials only, rather than to perform a subgroup classification.

We organize the paper as follows. In the second section, we perform a symmetry classification of the equation and a classification of one-dimensional subalgebras of the symmetry algebra. In the third section, using one- and two-dimensional subgroups we reduce the equation to (1+1)-dimensional partial differential equations and second-order ordinary differential equations and then discuss the integrability of the reduced ODEs. We summarize the results in the final section.

2. The symmetry group and its Lie algebra

The method for determining the symmetry group of a differential equation is straightforward and described in several books [12–14]. The symmetry algebra is realized by vector fields

$$V = \xi\partial_x + \eta\partial_y + \tau\partial_t + \phi\partial_u \tag{2.1}$$

where ξ, η, τ, ϕ are functions of x, y, t, u. These coefficients are to be determined from the invariance condition

$$\mathrm{pr}^3 V(E)|_{E=0} = 0 \tag{2.2}$$

where $E = 0$ is the equation under study and $\mathrm{pr}^{(3)}V$ stands for the third prolongation of the vector field (see, for example, [12] for the general prolongation formula). Here we are faced with a group classification problem that comprises the determination of the coefficient functions in such a way that the equation admits nontrivial symmetries. Requiring the symmetry condition (2.2) and solving an over-determined system of linear PDEs we have:

Case I. $s(t)$ arbitrary.
For any function $s(t)$ the symmetry algebra of (1.2) is an infinite-dimensional Lie algebra which we denote by L_P. A general element of L_P for an arbitrary $s(t) \neq$ constant is represented by

$$V = X(f) + Y(g) \tag{2.3}$$

$$X(f) = f(t)\partial_x + f'(t)\partial_u \tag{2.4a}$$

$$Y(g) = g(t)\partial_y - \frac{g'(t)}{2s(t)}y\partial_x - \left(\frac{g'(t)}{2s(t)}\right)' y\partial_u \tag{2.4b}$$

where $f(t)$ and $g(t)$ are arbitrary smooth functions and the primes denote time derivatives. The commutation relations are

$$\begin{aligned}
&[X(f_1), X(f_2)] = 0 \qquad [X(f), Y(g)] = 0 \\
&[Y(g_1), Y(g_2)] = X\left(\frac{1}{2s}(g_1' g_2 - g_1 g_2')\right)
\end{aligned} \tag{2.5}$$

where [,] stands for the Lie bracket. It is readily seen that the coefficients of the vector fields $X(f)$ and $Y(g)$ multiplying ∂_t are necessarily zero. This implies that the symmetry algebra does not have the structure of a Virasoro algebra. This stems from the fact that the equation under study is non-integrable. All known integrable equations in 2 + 1 dimensions have symmetry algebras of Virasoro type.

The vector fields $X(f)$ and $Y(g)$ can be integrated to obtain the Lie group of transformations. Thus, if $u(x, y, t)$ is any solution to equation (1.2), then so are

$$\tilde{u} = u(x - \epsilon f(t)y, t) + \epsilon f'(t) \qquad \epsilon \in \mathbb{R} \tag{2.6a}$$

and

$$\tilde{u} = u\left(x + \frac{1}{2}g'(t)\epsilon\left(y + \frac{\epsilon}{2}g(t)\right), y - \epsilon g(t), t\right) - \frac{1}{2}g''(t)\epsilon\left(y + \frac{\epsilon}{2}g(t)\right) \tag{2.6b}$$

respectively.

The symmetry algebra becomes larger when $s(t)$ has two specific forms:

Case II. $s(t) = \sigma t^{\alpha}$, σ = constant.
In this case, in addition to V in (2.3) we have the following basis element (a dilation):

$$D_\alpha = x\partial_x + \frac{(3+2\alpha)}{2}y\partial_y + 2t\partial_t - u\partial_u. \tag{2.7a}$$

Case III. $s(t) = \sigma e^{\alpha t}$, σ = constant.
In this case the symmetry algebra is represented by V in (2.3) further extended by the following additional element

$$T_\alpha = \partial_t + \frac{\alpha}{2}y\partial_y. \tag{2.7b}$$

When $s(t)$ is constant the symmetry algebra is even much larger.

Case IV. $s = \sigma$ = constant.
The symmetry algebra is L_p with two additional generators

$$D_0 = x\partial_x + \frac{3}{2}y\partial_y + 2t\partial_t - u\partial_u \qquad T_0 = \partial_t. \tag{2.8}$$

The non-zero commutators amongst D_α, T_α, $X(f)$ and $Y(g)$ are

$$\begin{aligned}
&[X(f), D_\alpha] = X(f - 2tf') \qquad [Y(g), D_\alpha] = Y\left(\frac{(3+2\alpha)}{2}g - 2tg'\right) \\
&[X(f), T_\alpha] = -X(f') \qquad [Y(g), T_\alpha] = Y\left(\frac{\alpha}{2}g - g'\right).
\end{aligned} \tag{2.9}$$

The Lie algebra L with a basis $X(f)$, $Y(g)$ and D_α or T_α can be written as a semi-direct sum

$$L = \{X(f), Y(g)\} \oplus_s S \tag{2.10}$$

where $S = \{D_\alpha\}$ or $S = \{T_\alpha\}$. For the last case we write

$$L = \{X(f), Y(g)\} \oplus_s \{D_0, T_0\}. \tag{2.11}$$

Restricting $f(t)$ and $g(t)$ to be linear polynomials we obtain obvious physical symmetries spanned by

$$\begin{aligned} X &\equiv X(1) = \partial_x & Y &\equiv Y(1) = \partial_y \\ B &\equiv X(t) = t\partial_x + \partial_u & R &\equiv Y(t) = -\frac{\sigma}{2} y\partial_x + t\partial_y \end{aligned} \tag{2.12}$$

which are space translations, Galilei transformations in the x direction and pseudo-rotations, respectively. The six-dimensional subalgebra $L_0 = \{T_0, X, Y, D_0, R, B\}$ corresponding to the constant coefficient Burgers equation is solvable and contains a five-dimensional nilpotent ideal (the Nilradical) $NR(L_0) = \{T_0, X, Y, R, B\}$ (see table 1).

Table 1. The commutator table for the physical subalgebra L_0.

	X	Y	T_0	B	R	D_0
X	0	0	0	0	0	X
Y	0	0	0	0	$-\frac{1}{2\sigma}X$	$\frac{3}{2}Y$
T_0	0	0	0	X	Y	$2T_0$
B	0	0	$-X$	0	0	$-B$
R	0	$\frac{1}{2\sigma}X$	$-Y$	0	0	$-\frac{1}{2}R$
D_0	$-X$	$-\frac{3}{2}Y$	$-2T_0$	B	$-\frac{1}{2}R$	0

2.1. Low-dimensional subalgebras of the symmetry algebra

In order to be able to perform symmetry reductions in a systematic way, we need to classify subalgebras of the infinite-dimensional algebras. We use the approach followed in [10] as an adaptation of the methods developed for the classification of subalgebras of the finite-dimensional algebras to infinite-dimensional ones. The difference is that we obtain differential conditions on the arbitrary functions labelling the group elements, rather than algebraic conditions on the parameters labelling the group elements of the finite-dimensional group. We present a classification of the one-dimensional subalgebras of the symmetry algebra into conjugacy classes under the adjoint action of the symmetry group. We do this individually for each algebra classified in section 2.

Case I. $s(t) =$ arbitrary $\neq$ constant.
Conjugating the general element $V = X(f) + Y(g)$ by $Y(G)$ and using the commutation relations (2.5) we obtain

$$\mathrm{Ad}\{\exp(\lambda Y(G))\}V = X\left(f - \frac{\lambda}{2s}(G'g - Gg')\right) + Y(g).$$

If we choose a function $G(t)$ to be defined by

$$G(t) = 2ag\int_0^t (sfg^{-2})(u)\,\mathrm{d}u + cg$$

where a and c are arbitrary constants, as the function labelling the element $Y(G)$ of the symmetry algebra and $\lambda = a^{-1}$ as the value of the parameter λ of the one-parameter subgroup associated with $Y(G)$, we see that if $g \neq 0$, V is conjugate to $Y(g)$, otherwise to $X(f)$.

Case II. $s = \sigma t^{\alpha}$, $\alpha \neq 0$.
Conjugating the general element $V = aD_\alpha + X(f) + Y(g)$ with $a \neq 0$ by $X(F) + Y(G)$ we obtain

$$\begin{aligned}\mathrm{Ad}\{\exp(\lambda X(F) + \mu Y(G))\}V &= aD_\alpha + X\Bigg\{f - \lambda a(F - 2tF') \\ &\quad - \frac{\mu a t^{-\alpha}}{2\sigma}(G'g - Gg') + \frac{a\mu^2 t^{-\alpha}}{2\sigma}(GG' - t(G'^2 - GG''))\Bigg\} \\ &\quad + Y\left\{g - \mu a\left(\frac{3+2\alpha}{2}G - 2tG'\right)\right\} \end{aligned} \tag{2.13}$$

where we have used the commutation relations (2.5) and (2.9). With the functions $F(t)$ and $G(t)$ and parameters λ, μ suitably chosen, $X(f)$ and $Y(g)$ in V gets transformed away and we are left with the result that V is conjugate to D_α. If $a = 0$ then V is either conjugate to $Y(g)$, $g \neq 0$ or to $X(f)$, $g = 0$.

Case III. $s = \sigma e^{\alpha t}$, $\alpha \neq 0$.
Similarly, conjugating the general element $V = aT_\alpha + X(f) + Y(g)$, $a \neq 0$ by $X(F) + Y(G)$ one can show that V is conjugate to T_α.

Case IV. $s = \sigma =$ constant.
Using similar arguments it is easily seen that the general element $V = aT_0 + bD_0 + X(f) + Y(g)$, $a, b \neq 0$ is conjugate to $T_0 + \lambda D_0$, $\lambda \neq 0$. For $a = 0$, V is conjugate to D_0, for $b = 0$ to T_0.

3. Symmetry reductions

3.1. Reductions to PDEs

The general method for performing the symmetry reduction using some specific subgroup G_0 of the full symmetry group is to first find the invariants of G_0 and rewrite (2.1) in terms of them. The invariants are obtained by solving the system of PDEs

$$X_i I(x, y, t, u) = 0 \qquad i = 1, \ldots, r$$

where $\{X_1, X_2, \ldots X_r\}$ is some basis for the Lie algebra of G_0.

Below we perform reductions of (1.2) by one-dimensional subalgebras.

I.1. Subalgebra $L_{1,1} = \{Y(g)\}$.
We use the substitution

$$u = W(\xi, t) - \frac{1}{4g}\left(\frac{g'(t)}{s(t)}\right)' y^2 \qquad \xi = x + \frac{g'y^2}{4sg}$$

and obtain the reduced PDE

$$W_t + WW_\xi - W_{\xi\xi} + \frac{g'}{2g}W + \rho(t) - \frac{s}{2g}\left(\frac{g'}{s}\right)'\xi = 0$$

where $\rho(t)$ is an arbitrary function of integration. For $\rho = 0$ and $g =$ constant this equation is the one-dimensional Burgers equation

$$W_t + WW_\xi - W_{\xi\xi} = 0.$$

I.2. Subalgebra $L_{1,2} = \{X(f)\}$.
The reduction formula and the reduced PDE are

$$u = W(y,t) + \frac{f'}{f}x, \qquad W_{yy} = -\frac{f''}{sf}.$$

Integrating we obtain an exact solution depending on three arbitrary functions of time

$$u = -\frac{f''}{2sf}\,y^2 + \frac{f'}{f}x + A(t)y + B(t)$$

where $A(t)$ and $B(t)$ are arbitrary functions.

Additional reductions occur when $s(t) = \sigma t^\alpha$ or $\sigma \mathrm{e}^{\alpha t}$.

II. Subalgebra $L_1 = \{D_\alpha\}$.
Invariance under D_α implies

$$u = t^{-1/2}W(\xi,\eta), \qquad \xi = xt^{-1/2}, \quad \eta = yt^{-(3+2\alpha)/4}$$

with W satisfying the reduced PDE

$$\left\{-\frac{1}{2}\xi W_\xi + WW_\xi - \frac{1}{2}W - W_{\xi\xi} - \frac{(3+2\alpha)}{4}\eta W_\eta\right\}_\xi + \sigma W_{\eta\eta} = 0. \tag{3.1}$$

III. Subalgebra $L_1 = \{T_\alpha\}$.
Invariance under T implies

$$u = W(x,\eta) \qquad \eta = y\mathrm{e}^{-\alpha t/2}$$

with W satisfying

$$\left(WW_x - W_{xx} - \frac{\alpha}{2}\eta W_\eta\right)_x + \sigma W_{\eta\eta} = 0. \tag{3.2}$$

3.2. Reductions to ODEs

One can further reduce the above-obtained PDEs (3.1) and (3.2) to ODEs by imbedding D_α and T_α into two-dimensional subalgebras of the symmetry algebra. To this end we commute D_α and T_α with an element $V = X(f) + Y(g)$ and invoke that they form a two-dimensional subalgebra. This requirement implies that the functions $f(t)$ and $g(t)$ are no longer arbitrary but take some specific forms. Since there exist two isomorphy classes of two-dimensional Lie algebras, Abelian and non-Abelian, we distinguish between two algebras:

II$_1$. Abelian subalgebra.

$$L_{2,1} = \left\{D_\alpha, \left(t^{(3+2\alpha)/4}\right) + \nu X\left(t^{1/2}\right)\right\}$$

where D_α has the form (2.7a) and ν is a constant. Invariance under the two-dimensional subalgebra $L_{2,1}$ gives the invariant solution

$$u = t^{-1/2}\,H(\rho) + \frac{\nu}{2}yt^{-(5+2\alpha)/4} + \frac{(3+2\alpha)(1+2\alpha)}{64}y^2t^{-(\alpha+2)}$$
$$\rho = xt^{-1/2} - \nu yt^{-(3+2\alpha)/4} + \frac{3+2\alpha}{16}y^2t^{-(3+2\alpha)/2}$$

where $H(\rho)$ satisfies the second-order ODE

$$H'' - HH' + \left(\frac{\rho}{2} - 1\right)H' + \frac{(1-2\alpha)}{8}H = \frac{(1+2\alpha)(3+2\alpha)}{32}\rho + C \tag{3.3}$$

where C is an integration constant. This ODE belongs to the polynomial class of equations of second order [15]. We have not been able to integrate this ODE for any value of the parameter

α, neither have we found a first integral of polynomial type. However, for $\alpha = -3/2$ equation (3.3) is reduced to the Riccati equation of first order

$$H' - \tfrac{1}{2}H^2 + \tfrac{1}{2}\rho H = c_0\rho + c_1. \tag{3.4}$$

By the substitution $H = 2(\hat{H} + \frac{\rho}{4})$, one obtains the normal form of (3.4)

$$\hat{H}' = \hat{H}^2 + S(\rho) \qquad S = -\frac{\rho^2}{16} + \frac{c_0}{2}\rho + \hat{c}_1 \qquad \hat{c}_1 = \frac{c_1}{2} - \frac{1}{4}. \tag{3.5}$$

This particular value of α is the only value for which (3.3) passes the Painlevé test which is often an indication for the solution of the equation to be expressed in terms of elementary functions or elliptic functions. Indeed, for a particular choice of arbitrary constants, namely when $\hat{c}_1 + c_0^2 = 0$, the general solution of the normalized Riccati equation (3.5) is expressible in terms of the modified Bessel functions of fractional order in the form

$$h(\rho) = \sqrt{\hat{\rho}}\left[k_1\, I_{1/4}\left(\frac{\hat{\rho}^2}{2}\right) + k_2\, I_{-1/4}\left(\frac{\hat{\rho}^2}{2}\right)\right] \qquad \hat{\rho} = \frac{\rho}{4} - c_0$$

where $\hat{H}$ is obtained from h through logarithmic differentiation as

$$\hat{H} = -\frac{h'}{h}.$$

Thus we have obtained an exact solution of the original equation invariant under the group generated by $\{D_{-3/2}, Y + \nu X(\sqrt{t})\}$.

II$_2$. Non-Abelian subalgebra.

$$L_{2,2} = \left\{D_\alpha,\ B + Y\left(t^{(5+2\alpha)/4}\right)\right\}$$

where $B = X(t) = t\partial_x + \partial_u$. Invariance under $L_{2,2}$ implies that the solution has the form

$$u = t^{-1/2}\, H(\rho) + yt^{-(5+2\alpha)/4} + \frac{(2\alpha+5)(2\alpha-1)}{32}\, y^2 t^{-(\alpha+2)}$$

$$\rho = xt^{-1/2} - yt^{-(3+2\alpha)/4} + \frac{2\alpha+5}{16} y^2 t^{-(3+2\alpha)/2}$$

with $H(\rho)$ satisfying a second order ODE of the same form as (3.3).

III$_1$. Abelian subalgebra.

$$L_{2,1} = \left\{T_\alpha, Y\left(\mathrm{e}^{\alpha t/2}\right) + \partial_x\right\}. \tag{3.6}$$

Invariance under (3.6) implies that the solution has the form

$$u = H(\rho) + \frac{\alpha^2}{16}\, y^2 \mathrm{e}^{-\alpha t}$$

$$\rho = x - y\mathrm{e}^{-\alpha t/2} + \frac{\alpha}{8} y^2 \mathrm{e}^{-\alpha t}$$

with $H(\rho)$ satisfying the second order ODE

$$H'' - HH' - H' - \frac{\alpha}{4}H = C + \frac{\alpha^2}{8}\rho.$$

For $\alpha = 0$ it is immediate to see that this equation has a first integral

$$H' - \tfrac{1}{2}H^2 - H = c_0\rho + c_1 \tag{3.7}$$

which is a Riccati equation. Again, this equation has an exact solution in terms of Bessel functions. This means that we have found a solution of the constant coefficient PDE invariant under $\{\partial_t, \partial_y + \partial_x\}$.

III$_2$. Non-Abelian subalgebra.

$$L_{2,2} = \left\{T_\alpha,\ Y\left(e^{(\alpha+2)/2t}\right) + X(e^t)\right\}. \tag{3.8}$$

Invariance under (3.8) implies that the solution is

$$u = H(\rho) + y e^{-\alpha t/2} + \frac{(\alpha^2-4)}{16} y^2 e^{-\alpha t}$$

$$\rho = x - y e^{-\alpha t/2} + \frac{(\alpha+2)}{8} y^2 e^{-\alpha t}$$

with $H(\rho)$ satisfying

$$H'' - HH' - \frac{\alpha+2}{4} H - \frac{\alpha^2-4}{8}\rho = C.$$

Once again, for $\alpha = -2$, following an integration we have the Riccati equation

$$H' = \tfrac{1}{2}H^2 + c_0\rho + c_1$$

as a first integral. Solving this equation provides us with the solutions of the original equation invariant under the two-dimensional subalgebra $\{\partial_t - y\partial_y,\ \partial_y + e^t(\partial_x + \partial_u)\}$. However, setting $\alpha = 2$ we have

$$H'' - HH' - H = C$$

which is not of the Painlevé type. This is to be expected because transforming from variables $(\rho, H(\rho))$ to $(R, G(R))$ by setting $R = H, G = H'$ reduces it to an Abel equation of the second kind

$$GG_R - RG(R) - R = C.$$

For other values of α, unfortunately we failed to solve it completely or to find a first integral.

4. Conclusions

The results of sections 2 and 3 can be summarized as follows:

- We investigated the group classification problem for the generalized (2+1)-dimensional Burgers equation.
- We found a classification of the one-dimensional subalgebras of the symmetry algebra under the adjoint (conjugate) action of the symmetry group. Next we constructed the two-dimensional subalgebras by using one-dimensional subalgebras.
- We obtained a classification of the reductions of the original equation with $s(t) = \sigma t^\alpha$ and $s(t) = \sigma e^{\alpha t}$ to lower-dimensional PDEs and to second-order ODEs.
- We showed that the reduced ODEs can be written in a unified manner

$$H'' - HH' + (B_0 + B_1\rho)H' + C_0 H + D_0 + D_1\rho = 0$$

  where B_1, B_0, C_0, D_1, D_0 are constants. For particular values of the parameters α in the coefficient function $s(t)$, they admit first integrals as Riccati equations whose solutions are expressible in terms of Bessel functions. On the other hand, we showed that the reduced ODEs pass the Painlevé test only for these values. This is the case when $s = \sigma t^{-3/2}$, $s = \sigma e^{-2t}$ or $s =$ constant.

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